Pergamon

0021–8928(95)00012-7

J. Appl. Maths Mechs, Vol. 59, No. 1, pp. 99–109, 1995

0021–8928/95 $24.00+0.00

THREE-DIMENSIONAL BOUNDARY LAYERS ON COMPLEX-SHAPED BODIES AT ANGLES OF ATTACK†

V. A. ALEKSIN and S. N. KAZEIKIN

Moscow

(*Received* 23 *February* 1994)

The flow and heat transfer in a three-dimensional boundary layer when a compressible gas flows around a model body of complex shape at angles of attack α up to 30°, and around a spherically blunted cone when $\alpha = 10°$ are investigated both on the divergence line in the plane of symmetry of the body and on the whole surface. Full and simplified formulations with the corresponding systems of equations are given. Use is made of analytic and numerical methods for calculating the equations of three-dimensional laminar and turbulent boundary layers and data concerning the external inviscid flow and the geometry of the body. Particular attention is paid to the following problems: analysis of the system of equations in cases where it can be simplified, the closure of the averaged boundary-layer equations under turbulent conditions, the choice of the system of curvilinear coordinates, and an analysis of the effect of the governing parameters on the appearance of "separation" domains, characteristic zones and the lines of divergence and convergence on the surface.

When a compressible gas flows around a complex-shaped body at an angle of attack, a three-dimensional cross-flow occurs in the boundary layer which, to a significant extent, determines the magnitudes of the local friction and the heat flux on the surface in a number of flow domains. The intensity of the secondary flows depends on the three-dimensional form of the configuration of the body and the value of the angle of attack. It can reach high values even in the case of small angles of attack, and the boundary layers which are formed are substantially three-dimensional. A detailed exposition of the formulations of a number of three-dimensional problems and of methods of solving them analytically and numerically for laminar and turbulent flows can be found in [1, 2].‡

1. FORMULATION OF THE PROBLEM

Making the conventional assumptions of boundary-layer theory and neglecting the effect of the normal Reynolds stresses, the system of equations for the mean characteristics of a three-dimensional boundary layer in a compressible flow of homogeneous perfect gas in a curvilinear system of coordinates ξ, η, ζ, associated with the surface of the body $\zeta = 0$ around which the flow is occurring, has the form

$$\frac{\partial}{\partial\xi}\left(\rho u\sqrt{\frac{g}{g_{11}}}\right)+\frac{\partial}{\partial\eta}\left(\rho w\sqrt{\frac{g}{g_{22}}}\right)+\sqrt{g}\,\frac{\partial}{\partial\zeta}(\rho\upsilon)=0$$

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial u}{\partial\xi}+\frac{w}{\sqrt{g_{22}}}\frac{\partial u}{\partial\eta}+\upsilon\frac{\partial u}{\partial\zeta}+A_1u^2+A_2w^2+A_3uw=\frac{A_4}{\rho}+\frac{1}{\rho}\frac{\partial}{\partial\zeta}\left(\mu\frac{\partial u}{\partial\zeta}-\rho\langle u'\upsilon'\rangle\right)$$

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial w}{\partial\xi}+\frac{w}{\sqrt{g_{22}}}\frac{\partial w}{\partial\eta}+\upsilon\frac{\partial w}{\partial\zeta}+B_1u^2+B_2w^2+B_3uw=\frac{B_4}{\rho}+\frac{1}{\rho}\frac{\partial}{\partial\zeta}\left(\mu\frac{\partial w}{\partial\zeta}-\rho\langle w'\upsilon'\rangle\right) \quad (1.1)$$

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial h}{\partial\xi}+\frac{w}{\sqrt{g_{22}}}\frac{\partial h}{\partial\eta}+\upsilon\frac{\partial h}{\partial\zeta}=\frac{1}{\rho}\frac{\partial}{\partial\zeta}\left[\frac{\lambda}{c_p}\frac{\partial h}{\partial\zeta}-\rho\langle h'\upsilon'\rangle\right]+$$

$$+\frac{u}{\rho\sqrt{g_{11}}}\frac{\partial p}{\partial\xi}+\frac{w}{\rho\sqrt{g_{22}}}\frac{\partial p}{\partial\eta}+\frac{\mu}{\rho}\left[\left(\frac{\partial u}{\partial\zeta}\right)^2+2\cos\psi_0\frac{\partial u}{\partial\zeta}\frac{\partial w}{\partial\zeta}+\left(\frac{\partial w}{\partial\zeta}\right)^2\right]-$$

†*Prikl. Mat. Mekh.* Vol. 59, No. 1, pp. 109–120, 1995.
‡Also see: ALEKSIN V. A. and SHEVELEV Yu. D., Numerical investigation of three-dimensional turbulent boundary layers. Methods of calculation. Preprint No. 147, Inst. Problem. Mekh., Akad. Nauk SSSR, 1980.

$$-\langle u'\upsilon'\rangle\left(\frac{\partial u}{\partial\zeta}+\cos\psi_0\frac{\partial w}{\partial\zeta}\right)-\langle w'\upsilon'\rangle\left(\frac{\partial w}{\partial\zeta}+\cos\psi_0\frac{\partial u}{\partial\zeta}\right)$$

$$p=\rho RT,\quad \partial p/\partial\zeta=0$$

$$(g_{33}=1,\ g_{31}=g_{32}=0,\ \cos\psi_0=g_{12}/\sqrt{g_{11}g_{22}},\ g=g_{11}g_{22}-g_{12}^2)$$

We have assumed that terms containing fluctuations in the density, viscosity and thermal conductivity are small compared with terms containing their mean values. The first equation of system (1.1) is the equation of continuity, the second and third are the equations for the momentum in a projection onto the curvilinear coordinates ξ and η, and the fourth equation is the heat flux equation. Here, ξ and η are directed along the surface and ζ is directed along the normal to the surface, ξ is measured along the generatrices of the body while η is measured from the windward plane of symmetry in an anticlockwise direction. The static pressure p is a function of ξ and η. In the calculations, it is assumed that $\mu=\mu_e(h/h_e)^\omega$, $\lambda=\lambda_e(h/h_e)^\omega$, $\omega=0.75$, u, w and υ are the longitudinal, transverse and normal components of the velocity in the ξ, η ζ system, g_{ij} are the components of the metric tensor, h is the enthalpy, ρ is the density, T is the temperature, μ and λ are the coefficient of viscosity and the thermal conductivity, c_p is the heat capacity at constant pressure, R is the gas constant, the subscript e refers to the outer edge of the boundary layer, w refers to the values at the wall and t refers to a turbulent regime. The coefficients A_i and B_i ($i=1,\ldots,4$) are determined by the geometry of the body surface and by the parameters of the external inviscid flow.

The following boundary conditions are imposed at the surface

$$\zeta=0,\quad u=w=0,\quad \rho\upsilon=(\rho\upsilon)_w=F(\xi,\eta)$$
$$F_1[\xi,\eta,h_w,(\partial h/\partial\zeta)_w]=0:\quad h_0/H_0=t_w(\xi,\eta)\quad\text{or}\quad q_w=q_w(\xi,\eta) \tag{1.2}$$

The conditions

$$\zeta\to\infty,\quad u\to u_e,\quad w\to w_e,\quad h\to h_e \tag{1.3}$$

are satisfied at the outer edge of the boundary layer where the distributions of the velocity components $u_e(\xi,\eta)$, $w_e(\xi,\eta)$ and of the enthalpy $h_e(\xi,\eta)$ are known. The latter are found from the equations of gas dynamics or determined using experimental data.

The initial conditions are specified in such a manner as to ensure the existence and uniqueness of the solution of the problem. For this purpose, it is necessary to specify the initial velocity and enthalpy profiles in a certain domain D. This domain of initial data must satisfy the condition that all of the characteristic directions along which perturbations are generated depart from it into the calculated flow domain, which is the zone of influence of the initial data domain (the Retz effect principle [3]). The flow beyond the limits of the zone of influence is not subject to calculation since other initial data have an action on it and it depends on e flow outside this zone [4]. In the case when there is a leading stagnation point, the initial conditions are determined from the solution in the neighbourhood of this point.

2. THE MODELLING OF TURBULENCE

We shall use the concept of turbulent viscosity. Isotropic transport coefficients, the basis of which is the postulate that the directions of the tangential stress vectors $\tau(\tau_1,\tau_2)$ and $G(\partial u/\partial\zeta,\partial w/\partial\zeta)$ coincide [5], are the most widely used. Different versions of models of the anisotropic coefficients are well known [6].

Below, we use a model of effective transport coefficients (see the second footnote on p. 99) where the total shear stresses τ_1, τ_2 along the ξ and η axes, which include the shear stresses and the Reynolds stresses $-\rho\langle u'\upsilon'\rangle$, $-\rho\langle w'\upsilon'\rangle$ and, also, the total heat flux q are defined as follows:

$$\tau_1=\mu\frac{\partial u}{\partial\zeta}-\rho\langle u'\upsilon'\rangle=\mu_{\Sigma,1}\frac{\partial u}{\partial\zeta},\quad \tau_2=\mu\frac{\partial w}{\partial\zeta}-\rho\langle w'\upsilon'\rangle=\mu_{\Sigma,2}\frac{\partial w}{\partial\zeta}$$
$$q=\frac{\lambda}{c_p}\frac{\partial h}{\partial\zeta}-\rho\langle h'\upsilon'\rangle=\frac{\lambda_\Sigma}{c_p}\frac{\partial h}{\partial\zeta}. \tag{2.1}$$

In introducing effective turbulent transport coefficients, it is assumed that the turbulent coefficient of viscosity is isotropic. It follows from this assumption that the effective longitudinal and transverse coefficients are equal: $\mu_{\Sigma,1} = \mu_{\Sigma,2} = \mu_\Sigma$, and the direction of the total tangential stress τ coincides with the direction of **G**. It also follows that the Prandtl mixing length L is a scalar function and is invariant under a coordinate transformation: $L_1 = L_2 = L$.

Extension of the Prandtl hypothesis to a three-dimensional boundary layer yields the extension of the Prandtl formula [5]

$$\mu_t = \rho L^2 |\mathbf{G}| \tag{2.2}$$

For a non-orthogonal, curvilinear system of coordinates, which is normally associated the body surface, we have

$$|\mathbf{G}| = \left[\left(\frac{\partial u}{\partial \zeta}\right)^2 + 2\cos\psi_0 \frac{\partial u}{\partial \zeta}\frac{\partial w}{\partial \zeta} + \left(\frac{\partial w}{\partial \zeta}\right)^2\right]^{1/2}$$

The coefficient of effective viscosity μ_Σ depends on the local Reynolds numbers Re_A ($\mathrm{Re}_A = \mu_t/\mu$) and Re_*

$$\mu_\Sigma = \phi_*\mu = \tfrac{1}{2}\mu\,\mathrm{Re}_A^{-1}\{\mathrm{Re}_A^2 - \mathrm{Re}_*^2 + [(\mathrm{Re}_*^2 - \mathrm{Re}_A^2)^2 + 4\,\mathrm{Re}_*^2\mathrm{Re}_A]^{1/2}\} \tag{2.3}$$

The effective thermal conductivity is defined by analogy with the effective coefficient of viscosity

$$\begin{aligned} \lambda_\Sigma &= \lambda\phi_* F(\mathrm{Re}_A, \mathrm{Re}_*, \mathrm{Pr}, \mathrm{Pr}_t) \\ F &= 1 + (\mathrm{Pr}/\mathrm{Pr}_t - 1)\mathrm{Re}_A^2\{\mathrm{Re}_*^2(1 + \phi_*\,\mathrm{Re}_A/\mathrm{Re}_*^2)\}^{-1} \end{aligned} \tag{2.4}$$

Laminar and turbulent Prandtl numbers are defined using the laminar and turbulent transport coefficients: $\mathrm{Pr} = C_p\mu/\lambda$, $\mathrm{Pr}_t = C_p\mu_t/\lambda_t$. The values of Pr and Pr_t are assumed to be constant in the numerical calculations: $\mathrm{Pr} = 0.7$ and $\mathrm{Pr}_t = 0$.

The mixing length L is determined by the empirical function

$$\begin{aligned} L &= \beta_*\delta\Phi(\beta_*, k, \zeta/\delta), \\ \Phi &= \left[1 - \exp\left(-\frac{2k}{\beta_*}\frac{\zeta}{\delta}\right)\right] \Big/ \left[1 + \exp\left(-0,75\frac{2k}{\beta_*}\frac{\zeta}{\delta}\right)\right] \end{aligned} \tag{2.5}$$

where $k = 0.4$ is the Karman constant, $\beta_* = 0.1$ and δ is the thickness of the boundary layer determined by the velocity profile U.

The magnitude of local critical Reynolds number Re_* depends on a number of parameters and, in particular, on the pressure gradient parameter P^+, the surface permeability parameter υ_w^+, the Mach number M_e and Reynolds number Re_θ

$$\begin{aligned} &\mathrm{Re}_* = \mathrm{Re}_*(\mathrm{Re}_\theta, P^+, \upsilon_w^+, M_e) \\ &P^+ = \nu\rho^{-1}\upsilon_*^{-3}(\mathbf{U}_e|\mathbf{U}_e|^{-1}\cdot\nabla p_e), \quad \upsilon_w^+ = \upsilon_w/\upsilon_*, \quad \upsilon_* = \sqrt{\tau_w/\rho} \end{aligned} \tag{2.6}$$

In this paper, we use relation (2.6) in the form which has been previously used when calculating heat transfer in two-dimensional flows when there are appreciable longitudinal pressure gradients and surface permeability

$$\mathrm{Re}_* = \mathrm{Re}_{*,0}\cdot\{1 + a[\upsilon_w^+ + bP^+/(1 + c\upsilon_w^+)]\}$$

where a, b and c are constants which are equal to 5.15, 5.86 and 5.00, respectively.

The number Re_θ is defined using U_e, ν_e and the momentum thickness θ_1

$$\mathrm{Re}_\theta = \frac{U_e\theta_1}{\nu_e}, \quad \theta_1 = \int_0^\infty \frac{\rho U}{\rho_e U_e}\left(1 - \frac{U}{U_e}\right)d\zeta \tag{2.7}$$

The quantity $\mathrm{Re}_{*,\theta}$ determines the transition from a laminar regime to turbulent conditions along the surface of the body. The dependence of $\mathrm{Re}_{*,\theta}$ on Re_θ which has been previously suggested (see the footnote on p. 99) is used.

In the general case $\mathrm{Re}_{*,\theta}$ is a function of many parameters which describe the transition process in the boundary layer. Among these, one can pick out such factors as the degree of turbulence of the free stream Tu_∞, the Mach and Reynolds numbers, the longitudinal pressure gradient and surface permeability parameters, and the temperature factor.

In the case of three-dimensional boundary flows, complex turbulence models contain a different number of additional equations [5–7]. The biparametric $K-\varepsilon$ model for the turbulence kinetic energy K and its rate of dissipation ε involves two equations like the model in [7] (with additional equations for the Reynolds stresses $-\rho\langle u'v'\rangle$, $-\rho\langle w'v'\rangle$. The three-parameter model consists of the equations for K and the Reynolds stresses τ_{t1}, τ_{t2}, and it is not assumed in this case that the turbulence characteristics have isotropic properties nor that the directions of the vectors τ and $\mathbf{G}$ are the same. Near-wall damping functions, similar to those used in algebraic models, are introduced into the coefficients of the equations of certain of the terms in order to describe the properties of the near-wall flow.

An analogy between the processes of heat and momentum transfer, which leads to the need to introduce the turbulent Prandtl number Pr_t, is used to model turbulent heat transfer. In a number of cases, there is a complex dependence of the distribution of Pr_t on the properties of the gas and the boundary conditions which can be taken into account using just a single additional equation for the turbulent heat flux $q_t = -\rho\langle h'v'\rangle$ which is similar to the equation introduced in [8] for a class of two-dimensional boundary-layer flows.

3. SIMPLIFIED FORMULATIONS OF THE PROBLEM

When solving laminar and turbulent three-dimensional boundary-layer problems, the question arises of how to find the boundary conditions on the divergence lines. Here, the problem reduces to solving a system of ordinary difference equation. The solution of the singularity of the critical point is the first step.

In the case of flow around a blunt body which has a plane of symmetry and the velocity vector of the free stream lies in this plane, a Cartesian system of coordinates is introduced with the Z-axis directed along the body and the XOZ-plane lying in the plane of symmetry of the body. The system of coordinates of the boundary layer, (ξ, η, ζ), is normally associated with the surface. The normal to the body surface has the coordinates $\mathbf{n}(\psi_1, \psi_2\, \psi_3)$ and the relationship between the Cartesian and boundary-layer systems of coordinates can be written in the form

$$x = r_w(\xi,\eta)\cos\eta + \psi_1\zeta, \quad y = r_w(\xi,\eta)\sin\eta + \psi_2\zeta, \quad z = \chi(\xi,\eta) + \psi_3\zeta \tag{3.1}$$

Here, $r_w(\xi, \eta)$ is the distance from the Z-axis to the surface of the body. If x^i Cartesian coordinates and ξ^i are the coordinates of the boundary-layer system, then expressions can be obtained from (3.1) for the components of the metric tensor g_{pq} in the boundary-layer system of coordinates in terms of the components in the Cartesian system of coordinates g'_{ij}

$$g_{pq} = g'_{ij}\frac{\partial x^i}{\partial \xi^p}\frac{\partial x^j}{\partial \xi^q} \tag{3.2}$$

In the neighbourhood of the plane of symmetry, which forms the divergence line when intersecting the body surface, the functions occurring in the system of equations of a three-dimensional boundary layer satisfy the condition $f(-\eta) = f(\eta)$. Consequently, they can be written in the form

$$f = f^0 + f^2\eta^2 + O(\eta^4) \quad (f = \rho, u, v, p, H, r_w, \chi) \tag{3.3}$$

The relationship $w(-\eta) = -w(\eta)$ must be satisfied in the case of the projection of the velocity vector onto the η-axis, and therefore

$$w = w^1\eta + w^3\eta^3 + O(\eta^5) \tag{3.4}$$

Using (3.2) and (3.3), the metric of the body surface in the neighbourhood of the plane of symmetry can be represented in the form

$$r_w(\xi,\eta) = r_w^0(\xi) + r_w^2(\xi)\eta^2 + O(\eta^4), \quad \chi(\xi,\eta) = \chi_0(\xi) + \chi_2(\xi)\eta^2 + O(\eta^4)$$

$$g_{11} = G + 2\left(\frac{dr_w^0}{d\xi}\frac{dr_w^2}{d\xi} + \frac{d\chi_0}{d\xi}\frac{d\chi_2}{d\xi}\right)\eta^2 + O(\eta^4) \tag{3.5}$$

$$g_{12} = 2\eta\left(\frac{d\chi_0}{d\xi}\chi_2 + \frac{dr_w^0}{d\xi}r_w^2\right) + O(\eta^3), \quad g_{22} = (r_w^0)^2 + O(\eta^2)$$

$$g = g_{11}g_{22} - g_{12}^2 = (r_w^0)^2 G + O(\eta^2), \quad G = \left(\frac{d\chi_0}{d\xi}\right)^2 + \left(\frac{dr_w^0}{d\xi}\right)^2$$

The coefficients A_i, B_i ($i = 1, 2, 3, 4$), occurring in the equations of motion of boundary-layer system (1.1), which depend on the metric are

$$A_1 = O(\eta^2)$$

$$A_2 = 2\left(\frac{d\chi_0}{d\xi}\chi_2 + \frac{dr_w^0}{d\xi}r_w^2\right)\left((r_w^0)^2\sqrt{G}\right)^{-1} + O(\eta) = A_2^0 + O(\eta), \quad A_3 = O(\eta)$$

$$B_1 = \frac{2\eta}{r_w^0 G}\left(\frac{dr_w^0}{d\xi}\chi_2 - \frac{d\chi_0}{d\xi}r_w^2\right)\left(\frac{dr_w^0}{d\xi}\frac{d^2\chi_0}{d\xi^2} - \frac{d\chi_0}{d\xi}\frac{d^2r_w^0}{d\xi^2}\right) = B_1^1\eta + O(\eta^2)$$

$$B_2 = O(\eta), \quad B_3 = \frac{dr_w^0}{d\xi}\Big/(r_w^0\sqrt{G}) + O(\eta) = B_3^0 + O(\eta)$$

$$A_4 = \frac{1}{\sqrt{G}}\frac{dp_0}{d\xi} + O(\eta^2), \quad B_4 = \frac{2\eta}{r_w^0}\left[\frac{1}{G}\left(\frac{d\chi_0}{d\xi}\chi_2 + \frac{dr_w^0}{d\xi}r_w^2\right)\frac{dp_0}{d\xi} - p_2\right] + O(\eta^2)$$

Consequently, the system of equations of the three-dimensional boundary layer (1.1) in the neighbourhood of the plane of symmetry can be written, after dropping the small terms in η, as follows:

$$\frac{\partial(\rho_0 u_0 r_w^0)}{\partial\xi} + \sqrt{G}\left[\rho_0 w_1 + r_w^0\frac{\partial(\rho_0 \upsilon_0)}{\partial\xi}\right] = 0$$

$$\frac{u_0\rho_0}{\sqrt{G}}\frac{\partial u_0}{\partial\xi} + \upsilon_0\rho_0\frac{\partial u_0}{\partial\zeta} = -A_4 + \frac{\partial}{\partial\zeta}\left(\mu_\Sigma\frac{\partial u_0}{\partial\zeta}\right)$$

$$\frac{u_0}{\sqrt{G}}\frac{\partial w_1}{\partial\xi} + \frac{w_1^2}{r_w^0} + \upsilon_0\frac{\partial w_1}{\partial\zeta} + B_3^1 u_0^2 + B_1^0 u_0 w_1 = \frac{B_4}{\eta\rho_0} + \frac{1}{\rho_0}\frac{\partial}{\partial\zeta}\left(\mu_\Sigma\frac{\partial w_1}{\partial\zeta}\right) \tag{3.6}$$

$$\frac{\rho_0 u_0}{\sqrt{G}}\frac{\partial H_0}{\partial\xi} + \rho_0\upsilon_0\frac{\partial H_0}{\partial\zeta} = \frac{\partial}{\partial\zeta}\left\{\frac{\mu_\Sigma}{\sigma_\Sigma}\left[\frac{\partial H_0}{\partial\zeta} + (\sigma_\Sigma - 1)\frac{\partial}{\partial\zeta}\frac{u_0^2}{2}\right]\right\}$$

We will now specify the pressure on the body surface using Newton's formula

$$p = (\mathbf{n}\cdot\boldsymbol{\upsilon}_\infty/|\boldsymbol{\upsilon}_\infty|)$$

For the components of the normal $\mathbf{n}(\psi_1, \psi_2\ \psi_3)$, we have

$$\psi_1 = \frac{1}{G_r}\left(\cos\eta + \frac{\partial r_w}{\partial\eta}\frac{\sin\eta}{r_w}\right), \quad \psi_2 = \frac{1}{G_r}\left(\sin\eta - \frac{\partial r_w}{\partial\eta}\frac{\cos\eta}{r_w}\right), \quad \psi_3 = -\frac{1}{G_r}\frac{\partial r_w}{\partial z}$$

$$G_r = 1 + \left(\frac{\partial r_w}{\partial z}\right)^2 + \left(\frac{1}{r_w}\frac{\partial r_w}{\partial\eta}\right)^2$$

The unit vector in the direction of the vector $\boldsymbol{\upsilon}_\infty$ has the components ($-\sin\alpha$, 0, $\cos\alpha$), where α is the angle of attack. Hence

$$p=\frac{1}{G_r}\left(-\frac{\partial r_w}{\partial z}\cos\alpha-\sin\alpha\cos\eta-\frac{\partial r_w}{\partial\eta}\frac{\sin\eta\sin\alpha}{r_w}\right)^2 \tag{3.7}$$

In the neighbourhood of the plane of symmetry

$$p=\left(\frac{\partial r_w^0}{\partial z}\cos\alpha+\sin\alpha\right)^2\left[1+\left(\frac{\partial r_w^0}{\partial z}\right)^2\right]^{-1}+p_2\eta^2+O(\eta^4)$$

In the case of the coordinates which have been introduced, the equality

$$\xi=\int_0 \sqrt{1+(\partial r_\upsilon/\partial z)^2}\,dz,\quad \eta=\operatorname{arctg}(y/x)$$

is satisfied on the body surface and hence

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial z}+\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial z}=\left[1+\left(\frac{\partial r_w}{\partial z}\right)^2\right]^{1/2}\frac{\partial}{\partial\xi}$$

Consequently

$$p_0=\left(\frac{dr_w^0}{d\xi}\right)^2\cos^2\alpha+2\cos\alpha\sin\alpha\frac{dr_w^0}{d\xi}\frac{d\chi_0}{d\xi}+\left(\frac{d\chi_0}{d\xi}\right)^2\sin^2\alpha \tag{3.8}$$

It follows from the equations of motion of system (3.6), when $\zeta\to\infty$ on the outer edge of the boundary layer, that

$$\begin{aligned}&u_0^e\frac{du_0^e}{d\xi}=-\frac{1}{\rho_0^e}\frac{dp_0}{d\xi}\\&\frac{u_0^e}{\sqrt{G}}\frac{dw_1^e}{d\xi}+\frac{(w_1^e)^2}{r_w^0}+B_1^1(u_0^e)^2+B_3^0u_0^ew_1^e=\frac{2}{\rho_0^er_w^0}\left[\frac{1}{G}\left(\frac{d\chi_0}{d\xi}\chi_2+\frac{dr_w^0}{d\xi}r_w^2\right)\frac{dp_0}{d\xi}-p_2\right]\end{aligned} \tag{3.9}$$

The solution of system (3.9) in the neighbourhood of the critical point $\xi=\xi_{cr}$ is sought in the form $u_0^e=a(\xi-\xi_{cr})$, $w_1^e=b(\xi-\xi_{cr})$. We then obtain

$$\begin{aligned}&a^2(\xi-\xi_{cr})=-\frac{1}{\rho_0^e}\frac{dp_0}{d\xi}\\&\frac{ab(\xi-\xi_{cr})}{\sqrt{G}}+(\xi-\xi_{cr})^2\left(\frac{b^2}{r_w^0}+B_1^1a^2+B_3^0ab\right)=\frac{2}{\rho_0^er_w^0}\left[\frac{1}{G}\left(\frac{d\chi_0}{d\xi}\chi_2+\frac{dr_w^0}{d\xi}r_w^2\right)\frac{dp_0}{d\xi}-p_2\right]\end{aligned} \tag{3.10}$$

Let us assume that the leading part of the body is a sphere, the centre of which lies in the Z-axis and that the critical point falls in the spherical domain. In this case, $\chi=R[1-\cos(\xi/R)]$, $r_w=R\sin(\xi/R)$, where R is the radius of the sphere.

$$p=\cos^2(\xi/R-\alpha),\ p_2=-\sin\alpha\sin(\xi/R)\cos(\xi/R-\alpha)$$

is obtained from (3.7) for the neighbourhood of the plane of symmetry of the sphere.

The system of equations (3.10) for a sphere takes the form

$$\begin{aligned}&a^2=2/(\rho_0^eR^2)\\&ab(\xi-\xi_{cr})+b^2\frac{(\xi-\xi_{cr})^2}{r_w^0}+\frac{ab}{R\operatorname{tg}\alpha}(\xi-\xi_{cr})^2=\frac{2\sin^2\alpha}{\rho_0^er_w^0}\end{aligned} \tag{3.11}$$

It is clear from the second equation that, when $\alpha \neq 0$, the magnitude of b is of the order of $c/(\xi - \xi_{cr})$. Consequently, the indeterminacy in the neighbourhood of the critical point is only solved when the origin of coordinates coincides with the critical point, that is, when $\alpha = 0$. In this case, the system has the solution $a = R^{-1}\sqrt{(2/\rho_0^e)}$, $b = 0$.

If a perfect gas flows around the body, then

$$a = R^{-1}\sqrt{(2+(\gamma-1)M_\infty^2)/(\gamma M_\infty^2)}$$

4. THE COMPUTATIONAL METHODS

A numerical method of calculating the equations of a three-dimensional boundary layer is used in order to estimate the thermal state of the surface of the body and to find the segments where the most intensive heat transfer occurs. The scheme in [9] is the foundation of this method. This ensures fourth-order accuracy with respect to the normal coordinate. In this case, the grid is specified as being non-uniform and depends on the structure of the turbulent boundary layer. Boundary conditions of a general form are used, without changing the order of accuracy of the integration and the uniformity of the computational algorithm. Depending on the intensity of reorganization of the flow, non-uniform integration intervals are also specified in directions tangential to the surface. A method of tracking the direction of the flow velocity is also used. Here, directed differences, which depend on the sign of the transverse component of the velocity w, are used. The positions of the divergence and convergence lines and the surfaces of flow separation are not known in advance and are determined during the computational process. Methods, involving algorithms in which conditions are imposed which take account of the direction of the velocity vector, have been described in [10].

When studying the flow around bodies of complex shape, the problem of setting up adequate mathematical model of the surface of the actual body arises. Here, the requirements concerning the description of the geometry vary depending on the particular problem. In problems of external aerohydrodynamics, the geometry of a body has certain general properties: it is possible to pick out a principle direction, that is, the direction of motion, and the body has a plane of symmetry.

In this paper, we use an analytical specification of the surfaces of bodies around which flows occur. In this case, the initial data are approximated by a set of elementary surfaces which are described by algebraic functions of not higher than the third order.† A finite number of parameters with a physical meaning (the sweep angle, width of the fuselage, etc.) determine the form of these functions. Smooth joining of the component elementary pieces of the surfaces is carried out automatically. Here, the metric of the surface, the Christoffel symbols and the normal are found the required degree of smoothness. The advantage of this approach is the minimum amount of computer memory and computer time required. A change in the parameters of the body does not present any particular difficulties.

The external shape of a model body, the surface of which has been constructed using the method described, is shown in Fig. 1. The geometry of a more complex body, which is defined by the values of 14 parameters, is shown in Fig. 2. The shape of the family of bodies being considered is characterized by a spherical bluntness which transforms into a conical surface. There may be such elements as a cabin on the leeward side and wings.

The choice of the curvilinear system of coordinates, which is normally associated with the surface of the body, is a key step in solving the equations of the three-dimensional boundary layer on bodies of complex shape. The possibility that the computational method will describe the flow pattern with sufficient accuracy largely depends on this. When investigating the flow around bodies with spatial configurations at angles of attack, the problem of obtaining such an adequate description of the flow properties by solving partial differential equations is complicated in view of the need to make the grid points denser in a number of flow domains where the functions are undergoing rapid changes.

Several systems of coordinates [2] are employed here for calculating the boundary-layer characteristics over the whole body surface. A system of coordinates associated with a spherical system of coordinates (R, θ, φ) the origin of which coincides with the centre of the spherical bluntness is used in the neighbourhood of the leading critical point. A system of coordinates, corresponding to a cylindrical system of coordinates, is introduced on the remaining part of the surface.

†KAZEIKIN S. N., SEMUSHKINA E. V. and SHEVELEV Yu. D., Some methods of calculating and visualizing the geometry of a complex shape. Preprint No. 286, Inst. Problem Mekh. Akad. Nauk SSSR, 1987.

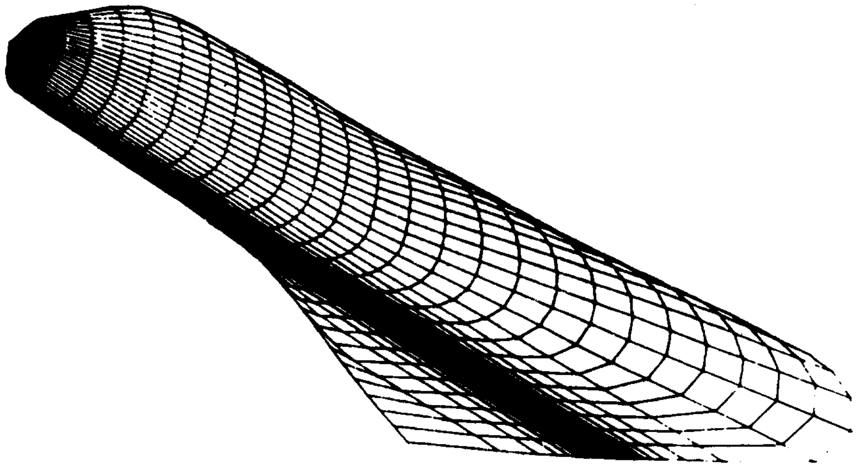

Fig. 1.

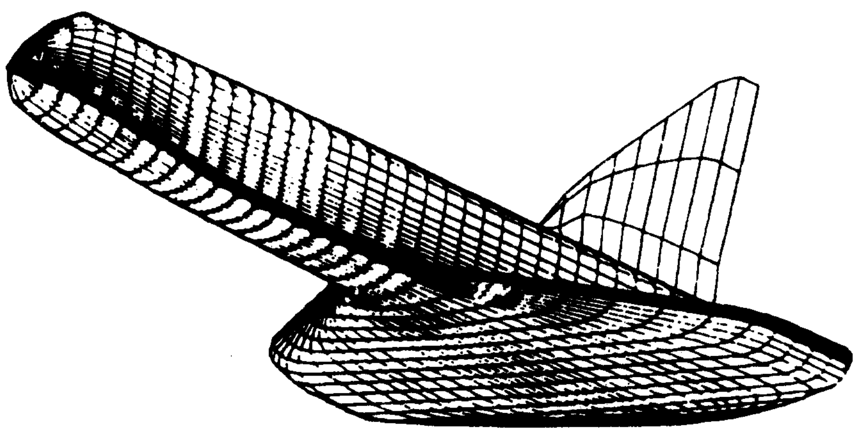

Fig. 2.

The solutions of the equations of the axially symmetric boundary layer serve as the initial conditions for calculating the equations of the three-dimensional boundary layer for the remaining part of the surface of bluntness. As the angle θ is increased, the shape of the body begins to differ from spherical, and the flow gradually acquires a spatial character. In order to calculate the flow and heat transfer parameters, it becomes necessary to use the three-dimensional boundary-layer equations for which a boundary-layer system of coordinates (θ^*, φ^*, ζ) is introduced, where $\theta^* = \theta$, $\varphi^* = \varphi$ and ζ is normal to the surface, the centre of which coincides with the centre of the spherical bluntness and the axis of which is inclined at a variable angle $\beta(\theta)$ to the Z-axis of the Cartesian system of coordinates (X, Y, Z). The system of coordinates (θ^*, φ^*, ζ) is directly associated with the rotating spherical system of coordinates (R, θ, φ).† The coordinate transformation formulae

$$\begin{aligned} x &= R(\theta^*,\varphi^*)\sin\theta^*\cos\varphi^*\cos\beta - R(\theta^*,\varphi^*)\cos\theta^*\sin\beta + \zeta\psi_1 \\ y &= R(\theta^*,\varphi^*)\sin\theta^*\sin\varphi^* + \zeta\psi_2 \\ z &= \xi_0(\theta)R(\theta^*,\varphi^*)\cos\theta^*\cos\beta^* + R(\theta^*,\varphi^*)\sin\theta^*\cos\varphi^*\sin\beta + \zeta\psi_3 \end{aligned} \tag{4.1}$$

are used to recalculate the components of the velocity vector of an inviscid flow from the system of coordinates (R, θ, φ) into the boundary-layer system of coordinates (θ^*, φ^*, ζ) with the specified functions $\xi_0(\theta)$, $\beta(\theta)$.

†POGORELOV N. V. and SHEVELEV Yu. D., Numerical investigation of supersonic flow around the leading section of blunt bodies at large angles of attack. Preprint No. 175., Inst., Problem Mekh. Akad. Nauk SSSR, 1981.

Here $R(\theta^*, \varphi^*)$ is the distance from the centre of the spherical system of coordinates to the surface of the body and ξ_0 is the origin of the centre of the system of coordinates (R, θ, φ).

5. RESULTS OF CALCULATIONS

The three-dimensional laminar boundary layer on the body shown in Fig. 1 was investigated for $M_\infty = 6$ and $\alpha = 5°$ using the analytic method of successive approximations [11] and a numerical method. The conditions at the outer edge were taken from calculations of the supersonic flow around bodies by an inviscid perfect gas. The nose section of the body was represented by a spherical bluntness. In the cross-section, two semi-ellipses are joined from the windward side by a straight line, that is, the bottom section of the body is planar. On the leeward side, there is a cabin and the wings are located at a sufficient distance from the origin.

The investigation enables us to draw a number of conclusions for bodies of similar type. Domains with an S-shaped velocity profile are distinguished. The divergence lines of the gas flow appear in the neighbourhood of the lateral edges, and the geometric lines of symmetry will not necessarily be divergence lines. Gas from this line spreads out to the leeward side and to the plane of symmetry over the whole thickness of the boundary layer. A change in the curvature of the body surface at the base of the wings causes a pronounced perturbation of the flow. On the leeward side, the flow around the cabin attracts attention. The streamlines, on bending around the protruding section of the cabin, converge to a single point. This gas flow is determined by the pressure distribution in the neighbourhood of the cabin.

The local maximum in the value of the coefficient of friction and the heat flux, which lies near the lateral edge, corresponds to a divergence line on the windward side. The large pressure drop, due to the occurrence of pronounced compression and rarefaction zones, scatters gas from the edge across the wing. The appearance of the local maximum in the coefficients of friction and heat flux on the wall is explained by this and, moreover, this maximum may exceed the local maximum close to the edge of the wing.

A comparison of the relative value of the heat transfer coefficient $(\mathrm{St}_\infty)_{\mathrm{rel}} = \mathrm{St}_\infty/\mathrm{St}_\infty\ (\xi = 1)$ obtained analytically (the solid lines) and by the numerical method (the dashed lines) along the coordinates lines $\eta = 80°$ (1) and $100°$ (2), located on the lateral edge is shown in Fig. 3. Good agreement of the results is obtained in the neighbourhood of the divergence line. On the other sections of the body, the analytical investigation is of a qualitative nature.

We also present the results of a numerical investigation of the flow and heat-transfer properties in the case of the compressible gas flow around a spherically blunted circular cone with an aperture angle $\theta_c = 10°$ at an angle of attack $\alpha = 10°$ when $M_\infty = 5.96$, $\gamma = 1.4$. The influence of a number of governing parameters on the development of three-dimensional flows can be analysed using the computational results obtained.

The boundary condition on the surface for the heat transfer equation was specified as

$$q_w + q_w^R = 0, \quad q_w^R = \varepsilon\sigma T_w^4 \tag{5.1}$$

that is, the re-radiation of energy on the surface occurs in accordance with the Stefan–Boltzmann law. The Boltzmann number $B_\infty = \rho_\infty \upsilon_\infty c_p/(\varepsilon\sigma T_\infty^3)$, which appears in boundary condition (5.1), is determined by the gas parameters in the free stream and the specified ε.

In the numerical calculations, the boundary condition (5.1) was linearized using Newton's method.

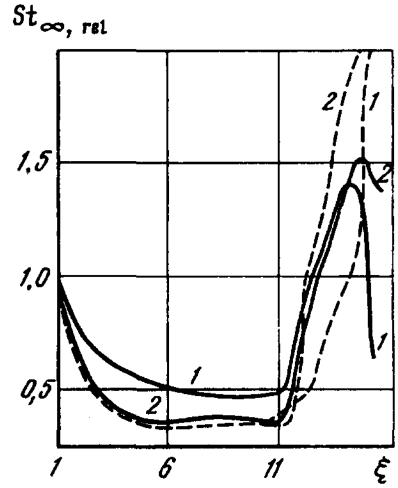

Fig. 3.

The external inviscid flow field was specified in accordance with the data presented earlier (see the footnote on p. 106). The distribution of the velocity components of this field was recalculated to yield the components of the boundary-layer system of coordinates. The field of the isobars on the surface of the body is shown in Fig. 4a–c as projections onto the vertical and horizontal planes and solely for those parts where the flow around the body is supersonic. Local pressure maxima occur on the windward divergence line in the plane of symmetry. In Fig. 4a the packing of the isobars on the windward side is noticeable as well as the decrease in its values on departing from this line onto the leeward side of the cone, where domains of minimum values are located in the plane of symmetry.

Numerical investigations were carried out for laminar and turbulent regimes at a height $H = 35$ km and $\varepsilon = 0.8$ for the case of an equilibrium surface radiation temperature T_{rw}, that is, condition (5.1). In the case of the laminar regime, the distributions of the isolines of the values of T_{rw} are monotonic on both the windward and leeward sides. The temperature falls as η increases from the windward plane of symmetry to the leeward plane of symmetry and the values of T_{rw} also decrease downstream along the flow. In the cross-section $Z = 10$, the difference between the temperature values of the windward and leeward sides increases as z increases and exceeds 120 K.

A version of the calculation, which corresponds to a turbulent flow with Reynolds number $\mathrm{Re}_\infty = 2.5 \times 10^7$ is shown in Fig. 5a–c: (a) shows the projections of the isolines of T_{rw} onto the vertical plane and (b) shows the projections onto the horizontal plane of the windward side and (c) shows the projections onto the horizontal plane of the leeward side. The values of T_{rw} differ from the corresponding values for the laminar regime by more than 200 K and, moreover, there is non-monotonicity in the distributions of T_{rw} and q_w on the windward side of the initial segment of the flow. The maximum values of T_{rw} occur on the spreading line which coincides with the line of symmetry of the windward side; the wall temperature on the leeward side is more than 200 K less than the values on the windward side.

The magnitude of the displacement thickness of the boundary layer δ^* characterizes the inverse effect of the layer on the external inviscid flow. In a number of cases, the value of δ_1^* can be determined in order to estimate the values of δ^*. On the windward side, it maintains an almost constant value, but on passing across the lateral edge, its values increase sharply. A general increase in the values of δ_1^* is noted on moving downstream for fixed values of η.

The greatest values of δ_1^* occur on the leeward side. According to the inviscid data, the greatest tapering of the entropy layer is noted in those zones where a minimum in the thickness of the boundary layer is observed. For instance, in the case of the computed values of the parameters when $\xi \geqslant 15.0$, there are segments in the flow where the displacement thickness of the boundary layer increases appreciably and it becomes necessary to take account of the vortex effect. The range of change in the parameters of the problem over which the entropy layer is swallowed by the boundary layer is determined from the calculated values of the boundary layer and the inviscid flow field [12, 13]. In the case of the specified parameters under consideration, the effect of the variability of the entropy along the outer edge of the boundary layer starts to have an effect at large distances from the bluntness ($z \geqslant 15$) and the vortex effect on the heat transfer falls off as the Reynolds number Re_∞ increases.

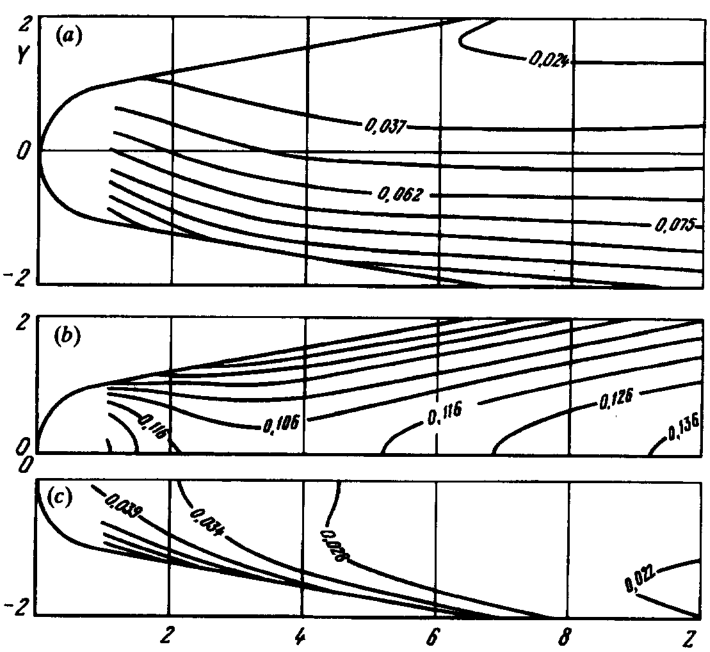

Fig. 4.

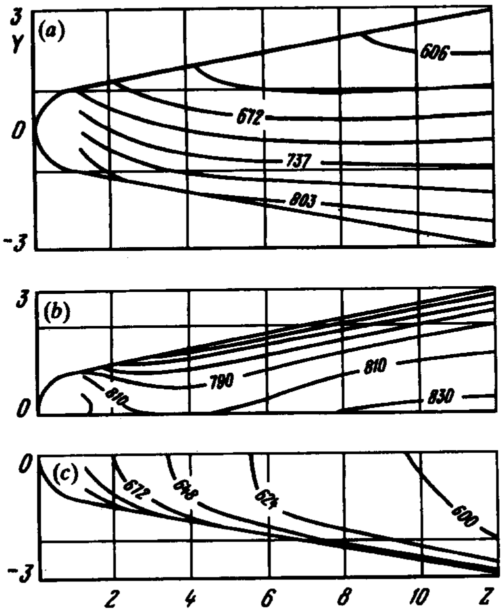

Fig. 5.

The distribution of the total local friction τ_w on the surface is characterized by the coefficient $C_{f,\infty} = 2\tau_w/(\rho_\infty v_\infty^2)$. The distributions of the values of $C_{f,\infty}$ along the lines γ = const as a function of ξ have a form similar to the distributions of T_{rw}. The greatest values of the local friction occur along a windward divergence line $\eta = 0$. Along the lateral edge $\eta = 80–90°$ and the changes in the friction τ_w are insignificant. On the leeward side of the cone the values of $C_{f,\infty}$ decrease monotonically downstream for all of the curves. Moreover, their fall off is the most intense in the plane of symmetry $\eta = 180°$. Here, the values of $C_{f,\infty}$ are minimal on the convergence line.

The results obtained show that the maximum values of the heat flux q_w and the local friction τ_w mainly occur along divergence lines and the minimum values correspond to the convergence lines on the body surface.

REFERENCES

1. ALEKSIN V. A. and SHEVELEV Yu. D., The three-dimensional turbulent boundary layer on a body of complex form. *Izv. Akad. Nauk SSSR, MZhG* 5, 25–25, 1986.
2. ALEKSIN V. A. and KAZEIKIN S. N., Heat transfer in a three-dimensional boundary layer accompanying the flow around a complex-shaped body at large angles of attack. In *The Gagarin Scientific Lectures on Cosmonautics and Aviation, 1987*, pp. 121–133. Nauka, Moscow, 1988.
3. HIRSHEL E. H. and KORDULLA W., *Shear Flow in Surface-oriented Coordinates* (*Notes on Numerical Fluid Mechanics*, Vol. 4).Vieweg, Braunschweig, 1981.
4. WANG K. C., On the determination of the zones of influence and dependence for three-dimensional boundary-layer equations. *J. Fluid Mech.* **48**, Pt 2, 397–404, 1971.
5. FEDYAYEVSKII K. K., GINEVSKII A. S. and KOLESNIKOV A. V., *Calculation of the Turbulent Boundary Layer of a Compressible Fluid*. Sudostroyeniye, Leningrad, 1973.
6. ROTTA J. C., A family of turbulence models for three-dimensional boundary layers. In *Turbulent Shear Flows*, Vol. 1, pp. 273–290. Mashinostroyeniye, Moscow, 1982.
7. BRADSHAW P., Calculation of three-dimensional turbulent boundary layers. *J. Fluid Mech.* **46**, Pt 3, 417–445, 1971.
8. MAKSIN P. L., PETUKHOV B. S. and POLYAKOV A. F., Calculation of turbulent heat transfer for a stabilized flow in tubes. In *Heat and Mass Transfer—V*, Vol. 1, pp. 14–24. Inst .Teplo-i Massoobmena, Minsk, 1976.
9. PETUKHOV I. V., Numerical calculation of two-dimensional flows in a boundary layer. In *Numerical Methods of Solving Differential and Integral Equations and Quadrature Formulae*, pp. 304–325. Nauka, Moscow, 1964.
10. FERNHOLZ H. H. and KRAUSE E. (Eds), *Three-dimensional Turbulent Boundary Layer*. IUTAM Symposium, 1982, Springer, Berlin, 1982.
11. TIRSKII G. A. and KOVACH E. A., The use of the method of successive approximations to integrate the boundary-layer equations. *Dokl. Akad. Nauk SSSR* **190**, 1, 61–64, 1969.
12. AVDUYEVSKII V. S., The effect of the curvature of a shock wave on the heat transfer in the three-dimensional supersonic flow around bodies. In *Investigation of Heat Exchange in Liquid and Gas Flows*, pp. 29–54. Mashinostroyeniye, Moscow, 1965.
13. YEL'KIN Yu. G., YERMAK Yu. N., LIPATOV I. I. and NEILAND V. Ya., Swallowing of the entropy layer on a blunt cone in the hypersonic flow of a viscous gas. *Uch. Zap. TsAGI* **14**, 1, 18–25, 1983.

Translated by E.L.S.